

Non-linear σ -model for odd triplet superconductivity in superconductor/ferromagnet structures

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(Dated: February 2, 2008)

We consider some properties of odd frequency triplet superconducting condensates. In order to describe fluctuations we construct a supermatrix σ -model for the superconductor/ferromagnet or superconductor/normal-metal structures. We show that an odd frequency triplet superconductor, when in isolation or coupled to a normal metal, generally displays behaviour comparable to a superconductor with the usual singlet pairings. However, for spin dependent systems such as the superconductor/ferromagnet the two types of superconductor have quite different behaviour. We discuss this difference by considering transformations under which the σ -model is invariant. Finally, we calculate the low energy density of states in a ferromagnet coupled to a singlet superconductor. If odd frequency triplet components are induced in the ferromagnet the density of states will have a micro-gap similar to the micro-gap found in normal metals coupled to a superconductor.

PACS numbers: 74.50.+r, 74.20.Rp, 73.23.-b

I. INTRODUCTION

The Pauli principle imposes important restrictions on the symmetry of the superconducting condensate in superconductors. The most common condensate is a singlet where the Cooper pairs have antiparallel spins (s - or d -wave). In this case, the wave function describing Cooper pairs is assumed to be invariant under the exchange of electron coordinates. Another possibility is a triplet pairing with the total spin of the pair equal to unity. In this case the wave function of the pair is assumed to change sign if the electrons exchange coordinates. The most famous example of the triplet pairing (p -wave) is superfluid He³ but triplet superconductivity has been recently discovered.^{2,3}

However, a characterization of the superconductor in terms of space symmetries of the wave function of Cooper pairs is somewhat oversimplified. The full information about the superconducting condensate is in fact given by an anomalous Green function (Gorkov function) $F(\epsilon)$. This function depends not only on the coordinates of the Cooper pair but also on the frequency ϵ . The previous discussion about the properties of the wave function of the Cooper pairs corresponds to the case when the condensate function $F(\epsilon)$ is an even function of the frequency ϵ although nothing forbids the function $F(\epsilon)$ from being an odd function of ϵ . If this alternative possibility were realized one would have a situation where the condensate function $F(\epsilon)$ is invariant under the permutation of electrons with triplet pairing but would change sign in the singlet case. So, odd condensate functions of frequencies allow, at least theoretically, p -wave singlets and s - and d -wave triplets.

In this paper we shall discuss some aspects of triplet Cooper pairings which are odd in frequency and even in momentum. A superconductor with an odd frequency triplet condensate was introduced by Berezinskii⁴ as a possible candidate for a phase of He³, though this was later found to not be the case. One may also consider other symmetry variations. For example, in Ref. 5 an odd singlet superconductor (one which is odd in both frequency and momentum) was discussed. Unfortunately, the authors of Refs. 4,5 did not find a microscopic model that would lead to the odd frequency condensate.

Recently, it was found that the odd triplet condensate can be induced in a superconductor/ferromagnet structure provided the magnetization in the ferromagnet is inhomogeneous.⁶ In this situation one does not need a special kind of an electron-electron interaction. It is sufficient that the ferromagnet is coupled to a standard singlet superconductor. This shows that, independent of whether the odd superconductivity can be obtained as the ground state of a microscopic model or not, a detailed study of its properties based just on the symmetry of the condensate may be of interest because it can be realized at least as a proximity effect.

In this paper we compare properties of the odd triplet superconductivity with those of the conventional singlet. We first consider a superconductor with odd frequency triplet pairings (S_t). We construct the Gorkov Green functions and write them in terms of an integral over supervectors, which allows us to obtain a supermatrix σ -model. It turns out that the form of the Green functions closely resembles those of a standard singlet superconductor (S_s). In fact, one can show that in many cases an S_t will have very similar properties to an S_s . Differences appear when one considers spin dependent structures such as a superconductor coupled to a ferromagnet (S_s/F or S_t/F). These two types of superconductors have different symmetries of the order parameter which leads to differences in the Josephson effect.

A qualitative discussion about the proximity effect in S_t/F structures may be made from considering transformations under which the σ -model is invariant. From these transformations one can determine which types of Cooper pairs are induced in the ferromagnet and whether the penetration is long-ranged or short-ranged. Generally, it is simpler to just solve the saddle point equation, but if the ferromagnet has a complicated inhomogeneous structure consideration of the transformational invariances may be useful.

It is well known that the density of states of an S_s in isolation (and also an S_t) has an energy gap equal to the value of the order parameter. A normal metal has no energy gap. However, in an S_s/N structure it has been shown that the density of states in the normal metal decreases quadratically at low energies and vanishes completely at zero energy.^{7,8} This region of vanishing density of states is called the ‘micro-gap’ and is a consequence of long-ranged Cooper pairs being induced in the normal metal. For most S_s/F structures no long-ranged Cooper pairs are induced in the ferromagnet and so there is no micro-gap comparable to the micro-gap found in S_s/N structures. However, as a result of some inhomogeneities in the ferromagnets (domain walls can be an example), an odd triplet condensate may be induced.⁶ We consider such a S_s/F structure and calculate the low energy C-mode fluctuations about the saddle point solution. From this we can calculate the density of states. We find that the density of states in the ferromagnet has a micro-gap similar to the micro-gap found in S_s/N structures. As concerns a S_t/F structure, the superconducting condensate will always penetrate the ferromagnet, even if the ferromagnet is homogeneous and this penetration will always be long-ranged. As a result, we would expect the ferromagnet in an S_t/F structure to always exhibit a micro-gap.

II. GORKOV GREEN FUNCTIONS

In this section we construct the Green functions for an odd frequency triplet condensate and compare it to the Green functions of an even frequency singlet. We begin with the general form of the superconductor Hamiltonian

$$H = \int dr \left[\psi_\alpha^\dagger(r) \mathcal{H}(r) \psi_\alpha(r) + \psi_\alpha^\dagger(r) V_{\alpha\beta}(r) \psi_\beta(r) + \frac{1}{2} \int dr' \psi_\delta^\dagger(r) \psi_\gamma^\dagger(r') U_{\delta\gamma\alpha\beta}(r, r') \psi_\alpha(r') \psi_\beta(r) \right] \quad (1)$$

where \mathcal{H} is the one-particle Hamiltonian, $V_{\alpha\beta}$ is the exchange field which may have some spatial dependence, $U_{\delta\gamma\alpha\beta}$ is the two-particle potential and ψ_ξ and ψ_ξ^\dagger are fermionic destruction and annihilation operators. This form of the Hamiltonian is completely general with regards to spin, time and position symmetries. Since H must be Hermitian $V = V^\dagger$ and $U = U^\dagger$. In coupled systems such as S/F and S/N the superconductor is defined to lie along the negative x -axis and the ferromagnet or normal-metal lies along the positive x -axis. The two-particle potential U is just defined in the superconductor so vanishes in F and N . The exchange field vanishes in S and N . From the above Hamiltonian and using the conventional mean field approximation we can construct the Green functions that have both normal and anomalous components. For more details see for example Ref. 9. The dynamic equations for ψ_ξ and ψ_ξ^\dagger are obtained from the identity $\frac{\partial \Phi}{\partial t} = i[H, \Phi]$. The Green function $G_{\alpha\beta}(X, X')$ and the anomalous Green function $F_{\alpha\beta}(X, X')$ are defined by

$$\begin{aligned} G_{\alpha\beta}(X, X') &= -\langle T \Psi_\alpha(X) \Psi_\beta^\dagger(X') \rangle, \\ F_{\alpha\beta}(r, r') &= -\langle T \Psi_\alpha(X) \Psi_\beta(X') \rangle, \\ F_{\alpha\beta}^\dagger(r, r') &= -\langle T \Psi_\alpha^\dagger(X') \Psi_\beta^\dagger(X) \rangle, \end{aligned} \quad (2)$$

where T is the time-ordering operator, $X = (r, t)$ and Ψ and Ψ^\dagger are the operators in the Heisenberg representation. Usually one complements the Green function equations with the self-consistency equation

$$\Delta_{\xi\alpha}(X, X') = U_{\xi\alpha\gamma\beta}(X, X') \langle \Psi_\gamma(X') \Psi_\beta(X) \rangle. \quad (3)$$

Because of the Pauli exclusion principle the anomalous Green functions F must be anti-symmetric under simultaneous position-time and spin exchange. It follows from Eq. (3) that the order parameter Δ has the same symmetry as the anomalous Green function F . In the singlet state S_s the order parameter is even in time and position exchange. In the triplet state S_t considered here the order parameter is odd in time exchange but even in position exchange. However, we emphasize that we cannot and do not try to present a microscopic model that would determine the odd triplet superconducting order parameter Δ but write it purely phenomenologically. Note that the odd triplet condensate function F can exist due to the proximity effect in S/F structures.⁶

We use the dynamic equations of Ψ_ξ and Ψ_ξ^\dagger and the definitions for the Green functions to write dynamic equations for the Green functions. In order to simplify the symmetry considerations we chose the order parameter to be $\Delta(X, X') = \Delta(r, t, t') \delta(r - r')$.¹⁴ For the conventional singlet superconductivity the function $\Delta(r, t, t')$ is invariant

under the exchange of t and t' whereas in the triplet case considered here it changes sign. After taking a Fourier transform the advanced and retarded Gorkov Green functions represented in particle-hole space are

$$\begin{pmatrix} \epsilon \pm i\delta/2 - \mathcal{H} - V & \Delta(x, \epsilon) \\ (-1)^{S+1} \Delta^*(x, -\epsilon) & -\epsilon \mp i\delta/2 - \mathcal{H} - V^* \end{pmatrix} \mathcal{G}^{R,A}(x, x', \epsilon) = \delta(x - x'),$$

$$\mathcal{G} = \begin{pmatrix} G & F \\ F^\dagger & G^\dagger \end{pmatrix}, \quad (4)$$

where S is the total spin of the Cooper pair and δ is a small positive real number, the sign in front of which determines the advanced or retarded nature of the Green function. We see that the difference between the equations for the conventional singlet and odd triplet superconductivities is minimal. Note that the spin dependence is hidden inside G , F , Δ and V .

If the spin is represented by the Pauli matrices σ we can expand the order parameter as $\Delta = \sum_{i=0}^3 \Delta_i \sigma_i$ and we may write each component in terms of a phase, $\Delta_i = |\Delta_i| e^{i\theta_i}$. We represent the triplet components of Δ by σ_0 , σ_1 and σ_3 and the singlet ones by σ_2 . With this choice we satisfy the symmetry relations $\Delta = -\Delta^T$ for the conventional singlet superconductivity and $\Delta = \Delta^T$ for the odd triplet. For conventional even frequency superconductors the order parameter is often assumed to be energetically independent, however, in the case of an S_t the order parameter must be odd in energy so we choose the simplest possibility $\Delta(x, \epsilon) = \text{sgn}(\epsilon) \Delta(x)$.

In order to study mesoscopic fluctuations we use the supersymmetry method.¹⁰ Within this technique one can write the solution of Eq. (4) in terms of a functional integral over supervectors ψ ^{8,10,11,12}

$$\begin{aligned} \mathcal{G}^{R,A}(x, x', \epsilon) &= i \int \psi_\alpha^{2,1}(x) \otimes \bar{\psi}_\alpha^{2,1}(x') \exp[-\mathcal{L}_{s,t}] \mathcal{D}\psi \\ \mathcal{L}_s &= i \int \bar{\psi}(y) \begin{pmatrix} \epsilon - i\delta\Lambda/2 - \mathcal{H} - V & \Delta(y) \\ -\Delta^*(y) & -\epsilon + i\delta\Lambda/2 - \mathcal{H} - V^* \end{pmatrix} \psi(y) dy \\ \mathcal{L}_t &= i \int \bar{\psi}(y) \begin{pmatrix} \epsilon - i\delta\Lambda/2 - \mathcal{H} - V & \text{sgn}(\epsilon) \Delta(y) \\ -\text{sgn}(\epsilon) \Delta^*(y) & -\epsilon + i\delta\Lambda/2 - \mathcal{H} - V^* \end{pmatrix} \psi(y) dy \end{aligned} \quad (5)$$

where $\mathcal{L}_{s,t}$ is the action for the singlet and the odd triplet superconductivity respectively and all other terms have the standard definitions. If we perform the gauge transformation $\psi \rightarrow \psi e^{i\frac{\tau}{4}[\text{sgn}(\epsilon)-1]\tau_3}$ and $\bar{\psi} \rightarrow \bar{\psi} e^{-i\frac{\tau}{4}[\text{sgn}(\epsilon)-1]\tau_3}$ where τ represents Pauli matrices of the particle-hole space we find that, if we ignore the spin dependence, the triplet action is no different from the singlet action but the coefficient of the exponential becomes $[\psi_\alpha^{2,1}(x) \otimes \bar{\psi}_\alpha^{2,1}(x')]_{mn} \rightarrow [\psi_\alpha^{2,1}(x) \otimes \bar{\psi}_\alpha^{2,1}(x')]_{mn} [\text{sgn}(\epsilon)]^{m-n}$ where m and n represent components of the particle-hole space. Thus, if spin is not important the normal odd triplet Green functions G are identical to the normal singlet Green functions but the anomalous triplet Green functions F differ from that of the singlet by a factor of $\text{sgn}(\epsilon)$, i.e., the singlet's anomalous Green functions are even in ϵ but the triplet's are odd, as expected from the initial symmetry requirements. As the normal Green functions determine the density of states the bulk singlet and the bulk triplet have the same density of states. Also, a S_t/N structure should be similar to a S_s/N structure since in these cases spin is not important.

III. TRANSFORMATIONAL INVARIANCES OF THE σ -MODEL

From Eq. (5) the construction of a σ -model is fairly straight forward. Using the standard method of derivation developed for the singlet superconductor the σ -model action may be shown to be^{8,11}

$$S = \frac{\pi\nu}{16} \text{str} \int [D(\partial Q)^2 + 4iQ(\epsilon\tau_3 - i\delta\Lambda\tau_3/2 - \tilde{\Delta} - \text{Re } V - i\tau_3\rho_3 \text{Im } V)]. \quad (6)$$

where ρ_3 is the third Pauli matrix in the time-reversal space, Q is a 32×32 supermatrix, ν is the bulk normal-metal density of states per spin and

$$\tilde{\Delta} = i\tau_2\rho_3[\sigma_0|\Delta_0| \exp(-i\theta_0\tau_3\rho_3) + \sigma_1|\Delta_1| \exp(-i\theta_1\tau_3\rho_3) + \sigma_3|\Delta_3| \exp(-i\theta_3\tau_3\rho_3)] - \sigma_2\tau_1\rho_3|\Delta_2| \exp(-i\theta_2\tau_3\rho_3). \quad (7)$$

The Q -matrices in Eq. (6) must satisfy as usual the charge conjugation symmetry and integrals with the action S must converge. In addition, one can find several transformations under which Q is invariant in the bulk superconductor (when $V = 0$).⁷ We define A to be invariant under the transformation C if $A = CA^TC^T$. Table I defines five transformations and the terms with which they are invariant. All the terms in the action of a triplet superconductor are invariant under the C_4 transform while the singlet superconductor action is invariant under the other four transforms.

TABLE I: Transformational invariances of the σ -model action. The matrix A is invariant under the transform C if $A = CA^T C^T$. The singlet superconductor action is invariant under the C_0 , C_1 , C_2 and C_3 transforms whereas the triplet superconductor action is only invariant under the C_4 transform. In addition both types of superconductor actions must have charge conjugation and convergence symmetry.

transform	invariance
$C_0 = i\tau_1$	$\sigma_2, \tau_1\sigma_2, \tau_2\sigma_2, \tau_3\sigma_{0,1,3}$
$C_1 = \tau_2\sigma_1$	$\sigma_3, \tau_1\sigma_{0,1,2}, \tau_2\sigma_{0,1,2}, \tau_3\sigma_{0,1,2}$
$C_2 = \tau_1\sigma_2$	$\sigma_{1,2,3}, \tau_1\sigma_{1,2,3}, \tau_2\sigma_{1,2,3}, \tau_3$
$C_3 = \tau_2\sigma_3$	$\sigma_1, \tau_1\sigma_{0,2,3}, \tau_2\sigma_{0,2,3}, \tau_3\sigma_{0,2,3}$
$C_4 = \tau_2$	$\sigma_2, \tau_1\sigma_{0,1,3}, \tau_2\sigma_{0,1,3}, \tau_3\sigma_{0,1,3}$

This appears to disagree with what was found in Ref. 7 where it was claimed that the singlet was invariant under the C_4 transform. The difference is due to the spin dependence of our σ -model. In general the ferromagnetic exchange field is of the form $V = h_0\sigma_0 + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$ (all the h_i must be real since $V = V^\dagger$). In the ferromagnet Q is not required to be invariant under any of the transforms in table I but they can help in determining the form of Q in the ferromagnet.

As an example on how to use the transformational invariances we consider an S_t/F structure with different exchange fields. The saddle-point equation of a superconductor σ -model is also known as the Usadel equation. The quasiclassical Green function which satisfies the Usadel equation is the saddle point solution of the σ -model and is represented by

$$g_0 = \begin{pmatrix} g & f \\ f^\dagger & g^\dagger \end{pmatrix} \quad (8)$$

in the particle-hole space with the constraint $g_0^2 = 1$. If we assume the temperature is just below the superconducting transition temperature or the tunneling resistivity is very large the Green function in the ferromagnet is $g^{A,R} = -g^{A,R\dagger} \sim \mp 1$. In this case the Usadel equation may be linearized and the retarded anomalous triplet Green function can be shown to satisfy

$$iD\partial_x^2 f - 2\epsilon f + Vf - fV^* = 0 \quad (9)$$

in the ferromagnet (having dropped the superscript). This is the same as the linearised equation in the ferromagnetic region of an S_s/F structure but, due to the boundary conditions, the spin structure of f must be different.^{6,13} The boundary conditions at the interface are

$$\begin{aligned} \partial_x f(0^+) &= [\rho(+)/R_b]f(0^-), & T \ll 1, \\ f(0^+) &= f(0^-), & T \sim 1, \end{aligned} \quad (10)$$

where ‘−’ is the superconducting side of the interface and ‘+’ is the ferromagnetic side, T is the transparency of the interface, $\rho(\pm)$ is the resistivity and R_b is the tunneling resistivity. As $x \rightarrow -\infty$ the Green function must approach the bulk superconductor solution and as $x \rightarrow \infty$ it must approach the bulk ferromagnet solution. Assuming that the proximity effect on the superconductor is small the well known bulk solution may be taken in the entire superconducting region $x < 0$ where $V = 0$ so $f(x < 0) = \text{sgn}(\epsilon)\Delta/\sqrt{\epsilon^2 - |\Delta|^2}$. This is the same solution as for a bulk S_s but with the extra term $\text{sgn}(\epsilon)$ which gives the required odd energy dependence and Δ has a different spin dependence. In the ferromagnetic region $x > 0$ the anomalous Green function is of the form $f(x > 0) = \sum_{i=0}^3 f_i(x)\sigma_i$ (assuming we have both triplet and singlet components). The boundary condition at $x \rightarrow \infty$ is that all the f_i must vanish.

If the magnetisation is of the form $V = h\sigma_j$, $j = 1, 2, 3$ then the solution of the linearised Usadel equation is that each f_i will exponentially decay. Two components will decay at a rate independent of the exchange field, κ_ϵ and the other two will decay at the rate $\kappa = \sqrt{\kappa_\epsilon^2 \pm \kappa_h^2}$ where $\kappa_\epsilon^2 = -2i\epsilon/D$ and $\kappa_h^2 = -2ih/D$. For example, if $V = h\sigma_3$ the σ_3 and σ_0 components of the anomalous Green function decay at the rate κ_ϵ while the σ_1 and σ_2 components decay at the rate κ . When h is large, as it generally is in such structures, the $\sigma_{0,3}$ components are long-ranged while the other two are short ranged. The boundary conditions at the interface require that the σ_2 component vanishes at the interface. Inducing long-ranged triplet components $\sigma_{0,3}$ in the ferromagnet of a S_t/F structure with exchange field $h\sigma_3$ should not be surprising. However, if $V = h\sigma_2$ we find that the σ_0 and σ_2 components decay rapidly at the rate κ and the σ_3 and σ_1 components decay slowly at the rate κ_ϵ . The boundary conditions at the interface will make the σ_2 component vanish at the interface. In contrast, boundary conditions in an S_s/F structure with a homogeneous

ferromagnet potential only allow the σ_2 anomalous component in the ferromagnet which always decays rapidly at the rate κ .

It is easiest to find which f_i will decay at which rate from the Usadel equation. However, one can also determine this from the transformational invariances of the σ -model. In more complicated cases it is easier to determine which components will be significant from the invariances rather than the Usadel equation, although the Usadel equation is required for quantitative results. One can show that the anomalous components which are invariant under the same transform as the σ -model will have the h dependent decay κ . For example, if we have S_t/F with $V = h\sigma_3$ the transforms under which the associated σ -model is invariant are C_1 and C_2 . These invariances are shared by $\tau_{1,2}\sigma_{1,2}$ so one may conclude that the σ_1 and σ_2 components of the anomalous Green functions decay at the h dependent rate κ . The other two anomalous components, σ_0 and σ_3 are not invariant under the C_1 and C_2 transformations so decay at the rate κ_ϵ which is independent of h . Similarly, if $V = h\sigma_2$ the action is invariant under the C_1 and C_3 transforms, as are the terms $\tau_{1,2}\sigma_{0,2}$. Therefore the $\sigma_{0,2}$ components of the anomalous Green functions are short-ranged, decaying at the rate κ , while the other two components $\sigma_{1,3}$ are long-ranged, decaying at the rate κ_ϵ .

One case of particular interest is when a superconductor is coupled to an inhomogeneous ferromagnet. It has been shown that at an S_s/F interface it is possible to induce both a singlet and an odd frequency triplet component in the ferromagnet if, for example, $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$.⁶ Here $\alpha = Ax$ for some constant A when $0 < x < w$ and $\alpha = Aw$ when $x > w$ where w is some positive constant. We shall briefly describe how the anomalous components induced in the ferromagnet may be determined from the transformational invariances of the action. At the interface the ferromagnet potential introduces the term $\tau_0\sigma_3$ into the action so at this point the action is invariant under the C_1 and C_2 transforms. As x increases a $\tau_3\sigma_2$ component appears in the action. Now the action is invariant only under the C_1 transform. Invariance under the C_1 and C_2 transforms at the interface implies short-ranged (decay is h dependent) anomalous components $\sigma_{1,2}$ and long-ranged (decay is h independent) components $\sigma_{0,3}$. However, as x increases we lose the invariance under the C_2 transform. When C_1 is the only transformational invariance the short-ranged components are $\sigma_{0,1,2}$ and only σ_3 is long-ranged. However, the boundary conditions cause the coefficient of the σ_3 component to vanish. We may conclude that, if the total rotation Aw is small the solution within the domain wall will be approximately similar to the solution at the S_s/F interface. Thus we would expect the σ_0 component to be long-ranged. If the rotation is increased the loss of invariance under the C_2 transform has a more significant effect on the range of the σ_0 component and it vanishes more rapidly. This result is shown in Fig. 2 of Ref. 6 in which the Usadel equation for this S_s/F structure was solved, however, due to a spin rotation of σ_1 the authors find the σ_1 component to be long-ranged.

IV. LOW ENERGY DENSITY OF STATES

The full solution of the σ -model is obtained by considering fluctuations about the saddle point solution. There are several different types of fluctuations which are relevant to different cases. The low energy C-mode fluctuations about the Usadel saddle-point solution are defined as being diagonal in the advanced-retarded space and are therefore quantum corrections to the Usadel solution. They have the further property that they are independent of the order parameter and any magnetic field. The C-modes dominate at energies below the Thouless energy D/L^2 where L is the length of the ferromagnet.⁸ We shall find the C-mode fluctuations for an S_s/F structure with $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$. We will then simplify our solution and provide a qualitative description of the low energy density of states. We are interested in seeing how the triplet component induced in the ferromagnet affects the density of states. Our method closely follows that of Ref. 8 where an S_s/N structure was considered.

If the solution of the Usadel equation is Q_U and we represent the C-mode corrections by the matrix T then the full solution of the supermatrix is $Q = TQ_U T^{-1}$. One can show⁸ that at very low energies the dominant C-mode is spatially constant, the so-called zero-mode. In addition, Q_U has a very slow spatial variation. The matrix Q must satisfy the convergence symmetry and the charge conjugation symmetry. The convergence symmetry is

$$Q = \eta Q^\dagger \eta^{-1}, \quad \eta = E_{11} \tau_3 \Lambda + E_{22} \quad (11)$$

and the charge conjugation symmetry is

$$Q = \tau Q^T \tau^{-1}, \quad \tau = E_{22} i \rho_2 + E_{11} \rho_1. \quad (12)$$

We have defined

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\text{bf}}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\text{bf}}, \quad (13)$$

where the subscript ‘bf’ indicates boson-fermion space. Since Q_U must also satisfy the above symmetries we may define the fluctuations as $T = e^W$ where W must satisfy

$$W^\dagger = -\eta W \eta^{-1}, \quad W^T = -\tau W \tau^{-1}. \quad (14)$$

The C-mode fluctuations must be insensitive to the superconducting order parameter and magnetic fields so we require

$$\begin{aligned} [W, \sigma_2 \tau_1 \rho_3], [W, \sigma_2 \tau_2] &= 0, & \text{the order parameter commutes through} \\ [W, \tau_3 \rho_3] &= 0, & \text{the magnetic field commutes through.} \end{aligned} \quad (15)$$

For a solution of W we may use the zero-mode derived in Ref. 8 but we must include some spin dependence

$$\begin{aligned} T &= v u a_1 a_2 a_3 \\ a_1 &= \exp(i \tfrac{1}{2} \theta_1 E_{22} \tau_1 \rho_1 \sigma_1) \\ a_2 &= \exp(i \tfrac{1}{2} \theta_2 E_{22} \tau_2 \rho_1 \sigma_2) \\ a_3 &= \exp(i \tfrac{1}{2} \theta_3 E_{22} \tau_1 \rho_2 \sigma_3) \\ u &= \exp(i y E_{22} \rho_3) \\ v &= \exp \begin{pmatrix} 0 & \lambda - \mu \rho_3 \\ \mu + \lambda \rho_3 & 0 \end{pmatrix}_{\text{bf}}, \end{aligned} \quad (16)$$

where y is some complex variable and λ and μ are Grassmann variables. The above solution is sufficiently general for our choice of V . Terms which satisfy the symmetry requirements and are not included in T are superfluous to our density of states calculation. We could have chosen, for example, spin dependent fluctuations with the matrix structures $E_{22} \tau_1 \rho_2 \sigma_1$, $E_{22} \tau_2 \rho_2 \sigma_2$ and $E_{22} \tau_1 \rho_1 \sigma_3$ as they also satisfy the symmetry requirements. However, they would add nothing extra to the final solution. The extra terms will either vanish or make a contribution identical to the one already obtained from $a_{1,2,3}$. One should note that the invariant transform of the action of a singlet superconductor coupled to a ferromagnet with $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$ is C_1 and that T is also invariant under the C_1 transform. If we chose a different exchange field, for example $V = h(\sigma_3 \cos \alpha + \sigma_1 \sin \alpha)$ we should choose a different form of T . The above choice of a_3 will not contribute to the action and should be replaced with $\exp(i \tfrac{1}{2} \theta_3 E_{22} \tau_1 \rho_1 \sigma_3)$. In this case the invariant transform of both the action and the fluctuations is C_2 . Deriving a suitable form of T can be quite tedious and the task is considerably shortened if one chooses T to have the same invariance transform as the action under consideration. As stated above, this will not give the most general form of T , but gives those parts which contribute uniquely to the density of states.

The solution of the Usadel saddle point equation is $Q_U = g_0$. One can show that the part diagonal in particle-hole space which describes the normal Green function is $g \tau_3$, i.e., $g = -g^\dagger$. The off-diagonals in particle hole space f and f^\dagger describe the anomalous Green function and may in general contain the terms τ_1 and $\tau_2 \rho_3$ multiplied by the spin components $\sigma_{0,1,3}$ and $\sigma_2 \rho_3$. The spin components which actually appear in the solution of Q_U will of course be dependent on the spin structure of the exchange field V . On substituting the general solution of Q_U with the fluctuations T into the action given in Eq. (6) with $V = h(\sigma_3 \cos \alpha + \sigma_2 \sin \alpha)$ one finds that all the anomalous components vanish. The singlet components vanish because they are proportional to the order parameter which commutes with T while the triplet components give zero supertrace. One can show this is true even with the most general form of T , which is why it is unnecessary to find the most general form. One finds the zero-mode action to be

$$S = -2i\tilde{s}(\cos \theta_1 \cos \theta_2 \cos \theta_3 - 1) + 2ih_1 \sin \theta_1 \sin \theta_2 \cos \theta_3 - 2ih_2 \sin \theta_1 \sin \theta_3 \cos \theta_2 \quad (17)$$

where

$$\begin{aligned} \tilde{s} &= \pi \epsilon \nu \int g dx \\ h_1 &= \pi h \nu \int g \cos \alpha dx \\ h_2 &= \pi h \nu \int g \sin \alpha dx. \end{aligned} \quad (18)$$

Since the C-modes are diagonal in the advance-retarded space we need only consider the retarded part so Q has been reduced to a 16×16 supermatrix and we may set $\delta = 0$. The density of states with C-type zero-mode fluctuations is

$$\begin{aligned} \rho &= \frac{\nu}{8} \text{Re} \langle \text{str} Q \sigma_3^{\text{bf}} \tau_3 \rangle \\ &= 2\nu \text{Re} g \langle 1 - \tfrac{1}{2}(1 - \cos \theta_1 \cos \theta_2 \cos \theta_3) + 2\lambda\mu(1 - \cos \theta_1 \cos \theta_2 \cos \theta_3) \rangle \end{aligned} \quad (19)$$

where the averaging is weighted by the action in equation (17) and we must perform a path integration over Q (which means an integral over the three θ 's, λ , μ and y). This form of the density of states and action is quite general and one would obtain something similar for any exchange field of the form $V = h(\sigma_i \cos \alpha + \sigma_j \sin \alpha)$.

As the path integration over all Q is fairly complex we shall make an approximation and assume that this solution will be at least qualitatively similar to the full solution. We approximate by setting two of the θ 's to zero. The result is the same no matter which θ we choose to be non-zero. In this simplified case our fluctuations are equivalent to those in Ref. 8 so we may assume the same Jacobian. The density of states can be shown to be

$$\begin{aligned} \rho &= 2\nu \operatorname{Re} \left[g \left(1 - \frac{1}{2} \int_0^\pi d\theta \sin \theta e^{2i\tilde{s}(\cos \theta - 1)} \right) \right] \\ &= 2\nu \operatorname{Re} \left[g \left(1 - \frac{\sin 4\tilde{s}}{4\tilde{s}} - \frac{1 - \cos 4\tilde{s}}{4i\tilde{s}} \right) \right]. \end{aligned} \quad (20)$$

This is the same formula as for an S_s/N structure, but the Usadel solution of an S_s/F is not the same as for an S_s/N so the value of \tilde{s} will be different. An exact solution of the Usadel equation for an S_s/F structure with a non-homogeneous exchange field does not exist.

We now calculate an approximate solution of the density of states in the region where the ferromagnet is homogeneous $w < x < L$. To find a solution for the density of states we use the approximate solution for the Usadel equation derived in Ref. 6 which is valid when the tunneling resistivity R_b from the superconductor to the ferromagnet is large (it is also valid near the phase transition but this requires $\Delta \ll \epsilon$ which does not satisfy our small energy requirement). In this limit the bulk solution may be taken in the superconductor $g = \epsilon / \sqrt{\epsilon^2 - |\Delta|^2}$ which is vanishingly small. In the ferromagnet limit we can set $g \sim 1$ and then the anomalous Green functions f can be calculated from the linearised Usadel equation (9). We may obtain a solution for g from the identity $g_0^2 = 1$ which implies $g = \sqrt{1 - f^\dagger f} \sim 1 - \frac{1}{2} f^\dagger f$. In Ref. 6 it is shown that f contains both a singlet component and a triplet component. If the ferromagnetic exchange field is large compared to the energy then the singlet part is much smaller than the triplet in the region $w < x < L$ so may be neglected. The coefficient of the triplet component is derived in Ref. 6 although some care must be taken as one must perform two rotations to make it compatible with the matrix structures used here. The result is, when taking just the triplet component, $f^\dagger f \sim -C^2$ where

$$C^{R,A} = \mp iAB(0) \sinh[\kappa_\epsilon(L-x)][\kappa_\epsilon \cosh \Theta_\epsilon \cosh \Theta_3 + \kappa_3 \sinh \Theta_\epsilon \sinh \Theta_3]^{-1} \quad (21)$$

for $w < x < L$ and where $B(0) = (\rho \xi_h / 2R_b) f_s$, $f_s = \Delta / \sqrt{\epsilon^2 - \Delta^2}$, $\Theta_\epsilon = \kappa_\epsilon L$, $\Theta_3 = \kappa_3 L$, $\kappa_3 = \sqrt{A^2 + \kappa_\epsilon^2}$. To calculate the action we require

$$\tilde{s} = \pi \epsilon \nu \left(\int_{-\infty}^0 g dx + \int_0^w g dx + \int_w^L g dx \right). \quad (22)$$

In the small energy limit g is very small in the superconductor so we will neglect the integral over negative x . If we assume that w is small then we may also neglect the second integral. So now \tilde{s} just depends on the value of g in the homogeneous part of the ferromagnet which we have found to be

$$g \sim 1 - \frac{1}{2} A^2 B(0)^2 \sinh^2[\kappa_\epsilon(L-x)] \left(\kappa_\epsilon \cosh L \kappa_\epsilon \cosh w \sqrt{A^2 + \kappa_\epsilon^2} + \sqrt{A^2 + \kappa_\epsilon^2} \sinh L \kappa_\epsilon \sinh w \sqrt{A^2 + \kappa_\epsilon^2} \right)^{-2} \quad (23)$$

and therefore

$$\begin{aligned} \tilde{s} &= \pi \epsilon \nu (L-w) \left(1 + \frac{1}{4} A^2 B(0)^2 \left(1 - \frac{1}{2} (L-w)^{-1} \kappa_\epsilon^{-1} \sinh[2\kappa_\epsilon(L-w)] \right) \right. \\ &\quad \times \left. \left(\kappa_\epsilon \cosh L \kappa_\epsilon \cosh w \sqrt{A^2 + \kappa_\epsilon^2} + \sqrt{A^2 + \kappa_\epsilon^2} \sinh L \kappa_\epsilon \sinh w \sqrt{A^2 + \kappa_\epsilon^2} \right)^{-2} \right). \end{aligned} \quad (24)$$

Substituting \tilde{s} into equation (20) gives the low energy density of states within the homogeneous part of the ferromagnet ($x > w$). We find that the density of states vanishes at $\epsilon = 0$ indicating the presence of a micro-gap and the low energy behaviour is quadratic in ϵ . This is similar to what has been found in an S_s/N structure.^{7,8} Note that in an S_s/F structure where there is no long-ranged anomalous Green function in the ferromagnet one would not expect a micro-gap in this high tunnelling resistivity limit. In this case $\tilde{s} \sim \pi \epsilon \nu L$ and as $L \rightarrow \infty$ the density of states is simply $\rho = 2\nu$, i.e., the bulk normal-metal density of states.

An equivalent calculation for an S_t/F structure is much simpler. The C-mode fluctuations are defined to commute with the order parameter so in the case of a triplet superconductor these fluctuations must be independent of spin. Therefore an S_t/F is similar to an S/N and one can show that equation (20), which is exact for S/N but an approximation for S_s/F , is exact for S_t/F . One can solve the linear Usadel equation in the ferromagnet to show that the form of the low energy density of states is the same as in the normal metal of a S/N structure, displaying a micro-gap as the energy vanishes.

V. CONCLUSION

We have considered an unusual type of triplet Cooper pairing which is defined by an order parameter which is even in the momentum (or position) and odd in the frequency (or time). It was found that, for the most part, a superconductor with odd triplet Cooper pairs is much like the standard singlet superconductor (even in position and time). In the bulk these superconductors would appear to be much the same, and also when coupled to a normal metal. The main difference between the two superconductors is their spin structure. Another difference is the energy dependence of the order parameter though, in many cases, this is not important.

If we consider a situation where the spin is unimportant we may obtain equations for S_t from equations for S_s by simply replacing the order parameter Δ with $\text{sgn}(\epsilon)\Delta$. However, in density of states calculations, for example, this change of sign is irrelevant. Where we do observe a difference between the S_t and the S_s is in cases where the spin is important. When an S_s is coupled to an inhomogeneous ferromagnet it is possible to induce a long-ranged triplet anomalous Green function component as well as a short-ranged singlet component in the ferromagnet. However, when an S_t is coupled to any type of ferromagnet a long-ranged triplet component always exists in the ferromagnet. One can determine which anomalous components will dominate the ferromagnet by considering the transformational invariances of the σ -model. We considered the low-energy fluctuations about the Usadel solution of an S_s/F structure with a non-homogeneous exchange field in order to see if the long-range triplet has a significant effect. We found that an S_s/F structure which induces a long-range anomalous component in F will have a density of states which vanishes quadratically at small energies. This micro-gap in the density of states is also expected in S_s/N and S_t/F structures but not in S_s/F structures which do not have any long-ranged anomalous components.

Acknowledgments

We are grateful to F. S. Bergeret and A. F. Volkov for useful discussions.

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